Isochronic potentials and new family of superintegrable systems

2004 J. Phys. A: Math. Gen. 37 4085

(http://iopscience.iop.org/0305-4470/37/13/011)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 4085–4095 PII: S0305-4470(04)67763-8

Isochronic potentials and new family of superintegrable systems

Cezary Gonera

Department of Theoretical Physics II, University of Łódź, Pomorska 149/153, 90-236 Łódź, Poland

Received 18 August 2003, in final form 22 January 2004
Published 17 March 2004
Online at stacks.iop.org/JPhysA/37/4085 (DOI: 10.1088/0305-4470/37/13/011)

Abstract
A general scheme for constructing superintegrable systems for separated Hamiltonians in an arbitrary number of degrees of freedom is presented. The resulting family contains previously known superintgrable systems with separated Hamiltonians (in Cartesian coordinates at least); however, in general, the models belonging to the family admit additional integrals which are nonpolynomial functions of momenta. An application of the method for the construction of superintegrable models of Liouville type is described.

PACS numbers: 03.20.+i, 02.90.+p

1. Introduction

In this paper we study some classical superintegrable systems. A Hamiltonian system of N degrees of freedom is called superintegrable if it is integrable in the Liouville sense [2] and admits, in addition to N Liouville integrals, some further globally defined ones. The maximal number of these additional integrals is $N-1$; in such a case we speak about maximal superintgrability.

Maximally superintegrable systems have many interesting properties. Their trajectories, in the confining region, are all closed; the choice of action-angle variables is not unique (in particular, there exists a set of action-angle variables such that the Hamiltonian is a function of one action variable only [11]); the separation of variables in the Hamilton–Jacobi (HJ) equation is, in general, possible in more than one way; they posses a common underlying symmetry [8, 13]. If the superintegrability property survives quantization, the energy spectrum is highly degenerate, the energy eigenspaces carrying a representation of nontrivial (non-Abelian) symmetry algebra.

The rather extensive list of known superintegrable systems is constantly enlarged. Many examples of superintegrable models have appeared as a result of systematic search concentrated on systems admitting integrals of motion at most second order in momenta [4, 6–8, 10, 23, 32]. It has been observed that this kind of 'quadratic superintegrability' is related to generalized symmetries [29] and exact solvability at quantum level [28]. Superintegrable systems with quadratic integrals have also been considered in spaces of nonzero constant curvature [21, 27] and of nonconstant curvature [19]. Most of these systems with quadratic integrals appear to live in low-dimensional phase space. Remarkable exceptions are provided by the Kepler problem, the Winternitz model [8] and its recent generalization [3] which are superintegrable for an arbitrary number of degrees of freedom. Although the list of superintegrable systems is in a sense dominated by rather well-understood and classified models with quadratic integrals it is in no way exhausted by them. Systems with higher order integrals have been studied in [5, 9, 16–18, 20, 31]. Again, most of them have been identified in low-dimensional phase space. Important exceptions are provided here by the harmonic oscillator with rational frequency ratio, Calogero model [1] (with harmonic term), Calogero–Moser [33] (without harmonic potential) and Sutherland models with hyperbolic potential [1, 14]. These superintegrable (in any dimension) models describing interacting particles on line admit integrals polynomial in momenta (except for the latter one).

The aim of this paper is to present some general scheme for constructing superintegrable models in an arbitrary number of degrees of freedom. We expand here considerably (as well as give more detailed arguments) the ideas contained in two previous notes [12, 15].

Our method does not assume any particular form of extra integrals from the very beginning. Nevertheless, it enables one to produce a variety of superintegrable systems admitting also integrals of higher order in momenta. Actually, in many cases these additional integrals appear to be so complicated that they cannot be computed analytically. We also use our scheme to construct superintegrable examples of the so-called Liouville Hamiltonians. The Liouville models (not to be confused with the notion of integrability in the Liouville sense) form the particular class of Staeckel systems which are general separable systems with quadratic integrals of motion (see, for example, [25]).

The paper is organised as follows. Section 2 is devoted to the study of isochronic potentials, i.e. those for which the period of motion is energy independent. We give the general construction of such potentials both for the whole real axis as well as the semiaxis. In section 3 these results are applied to construct superintegrable systems for completely separated Hamiltonians. The method of finding additional integrals of motion is outlined. The canonical transformation converting the superintegrable system into the set of independent harmonic oscillators is written out. In section 4 we discuss the application of the method to Liouville models. The explicit examples of such systems are constructed. In particular, we describe a superintegrable Liouville Hamiltonian which can be viewed as the deformation of a superintegrable harmonic one. Finally, section 5 is devoted to some conclusions. After completing the paper we became aware of [24] where a similar strategy was used to construct two-dimensional potentials admitting only closed orbits.

2. Isochronic potentials

Consider the one-dimensional motion on the whole real axis. We assume a potential $V(x)$ to be bounded from below and confining, for some energy interval at least. Without losing generality it can be assumed that zero is the absolute minimum of $V(x)$. Let us take into account a trajectory corresponding to some energy E.

The motion, being confined, is periodic. Let $T(E)$ be the period. In what follows we assume $T(E)$ to be continuously differentiable and nondecreasing in the confining region. Note that $V(x)$ can have no local minima in this region (except the global one). Indeed, if there exists some additional local minimum then there is also a local maximum in between. It corresponds either to the energy above the upper limit of confining energy interval or below it.

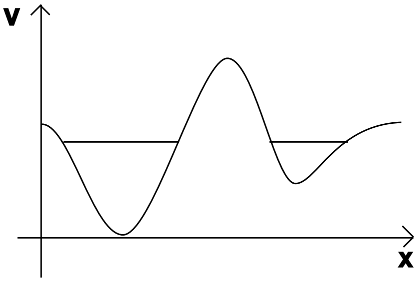

Figure 1. Accessible regions of motion for energy below the local maximum of potential, $V(x)$.

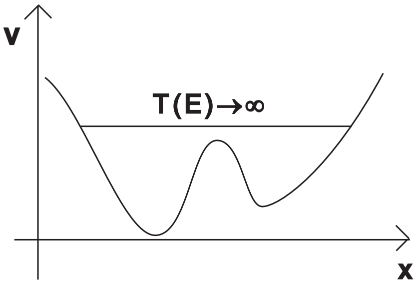

Figure 2. Accessible regions of motion for energy above the local maximum of potential $V(x)$.

In the former case we can consider two separated intervals of motion (figure 1) while in the latter $T(E)$ tends to infinity in the neighbourhood of the maximum, in contrast to the assumption concerning the behaviour of $T(E)$, (figure 2). Therefore, one can assume that the equation

$$V(x) = E \tag{1}$$

has exactly two solutions $x_{1,2}(E)$ in the whole confining interval (except $E = 0$).

With the above assumptions, given the period function $T(E)$ one can find all potentials $V(x)$ which produce $T(E)$ from the equation (see [22])

$$x_2(E) - x_1(E) = \frac{1}{\pi\sqrt{2m}} \int_0^E \frac{T(\varepsilon)\,\mathrm{d}\varepsilon}{\sqrt{E-\varepsilon}} \equiv f(E) \tag{2}$$

where it is assumed that $x_2(E) > x_1(E)$; $f(E)$ is continuously differentiable and increasing for $E > 0$.

All potentials satisfying (2) can be constructed as follows [12]. First, one can assume without losing generality that the minimum of $V(x)$ is attained at $x = 0$. For any $x < 0$ lying in the confining region we look for (unique) E such that $x = x_1(E)$ and define a function φ by

$$\varphi(x) = x_2(E). \tag{3a}$$

Similarly, for $x > 0$ we take $x = x_2(E)$ and

$$\varphi(x) = x_1(E). \tag{3b}$$

Finally,

$$\varphi(0) = 0.$$

Then φ is a one-to-one mapping of the confining interval onto itself satisfying

$$\varphi \circ \varphi = \mathrm{id}. \tag{4}$$

$V(x)$ can now be written in the form

$$V(x) = f^{-1}(|x - \varphi(x)|). \tag{5}$$

Note that $V(x)$ is differentiable provided f^{-1} and φ are differentiable. In fact, the only troublesome point is $x = 0$ where one can make the harmonic oscillator approximation, $f^{-1}(E) \sim E^2$. All φ satisfying equation (4) can be found [15]. In what follows we will be mainly interested in the case when the confining region extends to **R** or $\mathbf{R}_+$. In the former case we define

$$F(x) = \begin{cases} x & x \leqslant 0 \\ \varphi(-x) & x > 0 \end{cases} \tag{6}$$

then $F : \mathbf{R} \to \mathbf{R}$ is increasing, one-to-one and onto and

$$\varphi(x) = F(-F^{-1}(x)). \tag{7}$$

Note that F is not unique; in particular all odd functions F give $\varphi(x) = -x$.

Now, in order to deal with the potentials attaining their minimum at arbitrary point $x = a$ we should take φ such that $\varphi(a) = a$. Then $\varphi_1(x) \equiv \varphi(x + a) - a$ satisfies again $\varphi_1(0) = 0$ and the construction above applies. The counterpart of equation (7) reads

$$\varphi(x) = F(c - F^{-1}(x)) \qquad c = 2a. \tag{8}$$

Assume now that the confining region is $\mathbf{R}_+$. Then $V(x)$ takes the form

$$V(x) = f^{-1}(|x - \tilde{\varphi}(x)|) \tag{9}$$

where $\tilde{\varphi} : \mathbf{R}_+ \to \mathbf{R}_+$ is decreasing, one-to-one and onto. All such φ can be constructed by noting that the function

$$\varphi = \ln \circ \tilde{\varphi} \circ \ln^{-1} \tag{10}$$

is of the type considered previously. In particular, instead of equation (8) we obtain

$$\tilde{\varphi}(x) = \tilde{F}\left(\frac{\tilde{c}}{\tilde{F}^{-1}(x)}\right) \tag{11}$$

where $\tilde{F} : \mathbf{R}_+ \to \mathbf{R}_+$ is one-to-one and onto. From the point of view of superintegrable systems the most important is the case of constant T, $T(E) \equiv T$. Equations (2) and (5) imply then the following form of the potential:

$$V(x) = \frac{m\pi^2}{2T^2}(x - \varphi(x))^2. \tag{12}$$

With the help of equations (8) and (11) one can generate many examples of such 'isochronic' potentials. For example, with $\varphi(x) = \frac{1}{x}$ one obtains the Winternitz potential, while $\varphi(x)) = -x$ produces the harmonic oscillator. A simple choice of F, equation (8) or $\tilde{F}$, equation (11) can yield quite involved potentials. In particular, $F(x) = \alpha x^3$ gives

$$V(x) = \frac{m\pi^2}{2T^2}\left(\alpha c^3 - 3c^2\sqrt[3]{\alpha x} + 3c\sqrt[3]{x^2} - 2x\right)^2. \tag{13}$$

Another simple choice,

$$F(x) = a \sinh\left(\frac{x}{a}\right) \tag{14}$$

results in

$$V(x) = \frac{m\pi^2}{2T^2}\left(a \sinh\left(\frac{c}{a}\right)\sqrt{1 + \frac{x^2}{a^2}} - \left(1 + \cosh\left(\frac{c}{a}\right)\right)\frac{x}{a}\right)^2. \tag{15}$$

The latter example, which can be viewed as the one-parameter deformation of the harmonic oscillator, will be used in what follows.

It is worthwhile writing out the action-angle variables (J, Q) for our isochronic potentials. We have

$$J = \frac{1}{\pi}\int_{x_1(E)}^{x_2(E)} \sqrt{2m(E - V(x))}\,\mathrm{d}x. \tag{16}$$

Now, due to $\frac{\partial E}{\partial J} = \omega(J) \equiv \frac{2\pi}{T}$ ($T = \text{const}$) and our normalization of potentials, i.e. $V_{\min} = 0$, we obtain

$$J = \frac{TE}{2\pi} \equiv \frac{TH(x,p)}{2\pi}. \tag{17}$$

On the other hand, the standard formula gives

$$Q = m\omega \int_{x_1(E)}^{x} \frac{\mathrm{d}x}{\sqrt{2m(E - V(x))}} \equiv Q(x,E) \equiv Q(x, H(x,p)). \tag{18}$$

For the potentials related to $\varphi(x)$ or $\tilde{\varphi}(x)$ given by equations (8) and (11), respectively, equation (18) takes the form

$$Q = m\omega \int_{y_1}^{y} \frac{F'(y)\,\mathrm{d}y}{\sqrt{2m(E - (F(c-y) - F(y))^2)}} \qquad y = F^{-1}(x) \tag{19}$$

$$Q = m\omega \int_{y_1}^{y} \frac{\tilde{F}'(y)\,\mathrm{d}y}{\sqrt{2m\left(E - \left(\tilde{F}\left(\frac{\tilde{c}}{y}\right) - \tilde{F}(y)\right)^2\right)}} \qquad y = \tilde{F}^{-1}(x). \tag{20}$$

In general, the integrals (19), (20) cannot be computed analytically. From this point of view $F(x)$ given by equation (14) provides a nice exception (it belongs to the two-dimensional representation of the translation group). The relevant explicit formula reads

$$Q = \psi + \tanh\left(\frac{c}{2a}\right)\chi \tag{21}$$

where

$$\psi = \arcsin\left[A\left(\frac{x}{a}\cosh\left(\frac{c}{2a}\right) - \sqrt{1 + \left(\frac{x}{a}\right)^2}\sinh\left(\frac{c}{2a}\right)\right)\right]$$

$$A = \frac{1}{TE}\sqrt{2m}\pi a\cosh\left(\frac{c}{2a}\right)$$

$$\chi = \arcsin\left[B\left(\sqrt{1 + \left(\frac{x}{a}\right)^2}\cosh\left(\frac{c}{2a}\right) - \frac{x}{a}\sinh\left(\frac{c}{2a}\right)\right)\right]$$

$$B = \frac{\pi\sqrt{2m}a\cosh\left(\frac{c}{2a}\right)}{T\sqrt{E + \frac{2m\pi^2a^2}{T^2}\cosh^2\left(\frac{c}{2a}\right)}}.$$

Finally, let us note that one can construct potentials which are isochronic in some, given in advance, energy intervals.

3. Superintegrable models

Let us assume that forces related to an integrable system of N degrees of freedom are confining ones. Then one can introduce action-angle variables J_k, Q_k, $k = 1, \ldots, N$. The hypersurfaces of constant action variables are Arnold–Liouville tori.

The time dependence of angle variables is given by equations

$$\dot{Q}_k(t) = \omega_k(J) \equiv \frac{\partial H(J)}{\partial J_k}. \tag{22}$$

The form of trajectory strongly depends on whether there exist relations of the kind

$$\sum_{k=1}^{N} n_k w_k(J) = 0 \tag{23}$$

where n_k are integers which do not all vanish. If there is no such relation, the motion is ergodic, i.e. the trajectory covers the torus densely. This implies that there exists no global integral of motion functionally independent of action variables J_k. Indeed, if it existed the trajectory would belong to the intersection of the Liouville torus with the level hypersurface of this integral which is impossible if the trajectory is ergodic. To be more precise let us note that, in general, there can exist many tori in phase space for which equation (23) holds. It can even happen that they occupy a finite part of phase space; then the additional integrals of motion exist which are defined only over this domain (the way to construct particular examples of such models has been briefly sketched in [12]). On the other hand, if there is a dense subset of phase space where no relation (23) holds then the additional integrals of motion cannot exist.

In this paper we consider models for which the relations (23) are satisfied or not satisfied over the whole phase space. Therefore, the additional integrals, if they exist, are also defined over the whole phase space. Then, any relation of (23) type reduces by one the dimension of hypersurfaces to which the trajectories are confined. The related additional integral of motion is obtained by taking any periodic function (say, sine) of $\sum_{k=1}^{N} n_k Q_k$. The latter quantity is defined modulo an integer multiple of 2π which makes the integral well defined. Moreover, the Hamiltonian depends on $N-1$ variables; it is constant along the direction $\vec{n} = (n_1, \ldots, n_N)$ in the space of action variables. Such a model is called superintegrable. The maximal number of independent relations (23) is $N-1$. In this case there are $N-1$ additional independent integrals of motion constructed as above, the trajectories are closed (because all ratios $\frac{\omega_k(J)}{\omega_n(J)}$ are rational numbers) and the Hamiltonian is of the form

$$H(J) = H\left(\sum_{k=1}^{N} l_k J_k\right) \qquad l_k \in \mathbb{Z}. \tag{24}$$

Consider now the particular case of the separated Hamiltonian

$$H(J) = \sum_{k=1}^{N} H_k(J_k). \tag{25}$$

For H maximally superintegrable, equations (24) and (25) imply

$$H(J) = \alpha \sum_{k=1}^{N} l_k J_k \tag{26}$$

i.e. all frequencies are constant. This can also be seen from equation (23) if one takes into account that ω_k is a function of J_k only.

The simplest systems corresponding to (25) are provided by the Hamiltonians

$$H = \sum_{k=1}^{N} \left(\frac{p_k^2}{2m_k} + V_k(x_k)\right). \tag{27}$$

They are obviously integrable but generically not superintegrable. According to the above discussion in order to get a maximally superintegrable system all V_k should be isochronic with all ratios of periods $\frac{T_k}{T_l}$ being rational numbers, i.e. $T_k = n_k T$, $n_k \in \mathbb{Z}$, $T = \text{const}$.

Therefore, the general maximally superintegrable system of (27) type is given by

$$H = \sum_{k=1}^{N} \left(\frac{p_k^2}{2m_k} + \frac{m_k \pi^2}{2n_k^2 T^2}(x_k - \varphi_k(x_k))^2\right) \tag{28}$$

where functions φ_k are constructed as in section 2 (see equations (8) and (11)).

Taking into account equations (16) and (18) we can write out immediately the action-angle variables for this model

$$J_k = \frac{1}{\pi}\int_{x_{1k}(E_k)}^{x_{2k}(E_k)} \mathrm{d}x\sqrt{2m_k(E_k-(x-\varphi_k(x))^2)}$$
$$Q_k = m_k\omega_k\int_{x_{1k}(E_k)}^{x_k}\frac{\mathrm{d}x}{\sqrt{2m_k(E_k-(x-\varphi_k(x))^2)}}. \tag{29}$$

Now, in order to express these angle-action variables in terms of original canonical ones one has to replace (once the integrals on the RHS of equation (29) have been taken) E_k by H_k. In general, this results in very complicated functions of canonical variables; in particular these functions are not polynomial in momenta. Therefore, one cannot expect the additional integrals of motion (superintegrals), being periodic functions of linear combinations of angle variables, to be polynomial in momenta (see, for example, equations (14), (15) in [15]). Finally, let us note that the canonical transformation

$$\tilde{x}_k = \sqrt{\frac{n_kT}{\pi m_k}J_k(x,p)}\cos Q_k(x,p) \qquad \tilde{p}_k = \sqrt{\frac{4\pi m_k}{n_kT}J_k(x,p)}\sin Q_k(x,p) \tag{30}$$

converts system (28) into the set of independent oscillators

$$\tilde{H} = \sum_{k=1}^{N}\left(\frac{\tilde{p}_k^2}{2m_k} + m_k\frac{2\pi^2}{n_k^2T^2}\tilde{x}_k^2\right). \tag{31}$$

However, this does not mean that (28) is in any sense trivial. In fact any integrable system with a Hamiltonian given by a linear combination of action variables can be transformed into the set of independent oscillators as soon as one knows the action-angle variables in terms of the original canonical ones. Actually, the problem of finding isochronic systems given by equation (28) is equivalent to that of finding canonical transformations which transform superintegrable harmonic oscillators into natural (i.e. kinetic energy + potential) Hamiltonians; it is the very form of the Hamiltonians which makes the problem nontrivial.

4. Superintegrable Liouville models

The results of the previous section can be used to construct more general superintegrable models.

Consider the Liouville system defined by the Hamiltonian [25]

$$H = \frac{\sum_{k=1}^{N}\left(\frac{p_k^2}{2m_k} + u_k(x_k)\right)}{\sum_{k=1}^{N}s_k(x_k)}. \tag{32}$$

It is separable; the Hamilton–Jacobi equation separates into

$$\frac{p_k^2}{2m_k} + (u_k(x_k) - Es_k(x_k)) = \varepsilon_k \tag{33}$$

$$\sum_{k=1}^{N}\varepsilon_k = 0. \tag{34}$$

Therefore, the total energy E appears now as a parameter in the equivalent completely separated Hamiltonian

$$\tilde{H} = \sum_{k=1}^{N}\left(\frac{p_k^2}{2m_k} + (u_k(x_k) - Es_k(x_k))\right) \tag{35}$$

while there is an additional constraint (34) on partial energies ε_k. Assume we have found the action-angle variables $(\tilde{J}_k, \tilde{Q}_k)$ for the Hamiltonian $\tilde{H}$. Then we obtain the following relations:

$$\varepsilon_k = \varepsilon_k(\tilde{J}_k, E). \tag{36}$$

The constraint (34) determines E in terms of action variables $\tilde{J}_k$:

$$\sum_{k=1}^{N} \varepsilon_k(\tilde{J}_k, E) = 0. \tag{37}$$

Taking derivatives with respect to $\tilde{J}_l$ we obtain

$$\frac{\partial \varepsilon_l}{\partial \tilde{J}_l} + \left(\sum_{k=1}^{N} \frac{\partial \epsilon_k}{\partial E} \right) \frac{\partial E}{\partial \tilde{J}_l} = 0. \tag{38}$$

Note that $\frac{\partial E}{\partial \tilde{J}_l}$ is the actual frequency of the Liouville Hamiltonian (32) while $\frac{\partial \varepsilon_l}{\partial \tilde{J}_l}$ is the frequency corresponding to the lth component of the auxiliary Hamiltonian (35); these frequencies differ by the common factor $\sum_{k=1}^{N} \frac{\partial \varepsilon_k}{\partial E}$. We conclude that the solutions of energy E of the system (32) are a time reparametrization of zero energy solutions of (35). This can also be directly checked by using equations of motion. The relation between both models can be described as follows. The trajectories of the system (32) for a given E are identical to zero energy trajectories of (35) (where E enters as a parameter in the potential). In particular, the former are periodic if and only if the latter are. Therefore, if (35) is maximally superintegrable for any value of parameter E then all trajectories of (32) are periodic, i.e. (32) is also maximally superintegrable. So the problem reduces to that of finding isochronic potentials of the form

$$V_k(x) = u_k(x) - E s_k(x). \tag{39}$$

One obvious choice is to take the Winternitz potential with the energy E being identified with coupling constant in front of the $\frac{1}{x^2}$ term; more precisely, we put

$$\begin{aligned} u_k(x) &= \frac{m_k \omega_k^2 x^2}{2} \\ s_k(x) &= -\frac{\gamma_k^2}{x^2}. \end{aligned} \tag{40}$$

Then $V_k(x)$ is, for $E > 0$, the Winternitz potential with the period $T_k = \frac{\pi}{\omega_k}$. Note that this can be reversed, i.e. one takes

$$\begin{aligned} u_k(x) &= \frac{\delta_k^2}{x^2} \\ s_k(x) &= \frac{-\eta_k^2 x^2}{2}. \end{aligned} \tag{41}$$

Again, we obtain the Winternitz potential; this time the period depends on E (which plays here the role of a parameter and not the energy of auxiliary problem (35)), $T_k = \frac{\pi}{\eta_k}\sqrt{\frac{m_k}{E}}$. The E-dependence of T_k does not spoil superintegrability as long as the ratios of $\frac{\sqrt{m_k}}{\eta_k}$ are rational numbers. One can go even further and consider both u_k and s_k to be Winternitz potentials with different coefficients. Again things can be arranged to achieve superintegrability, although the conditions on coefficients are now more restrictive.

Another obvious possibility is to simply take $u_k(x)$ to be harmonic and $s_k(x)$ to be linear or vice versa. Finally, we could put $u_k(x) = s_k(x)$ to be any isochronic potential.

To see that less obvious possibilities exist consider the potential given by equation (15). It can be written in the form

$$V(x) = \gamma(E)(u(x) - Es(x) + u_0(E)) \tag{42}$$

where

$$\begin{aligned} E &= \frac{1}{T^2} 2m\pi^2 a^2 \tanh\left(\frac{c}{a}\right) \\ \gamma(E) &= \frac{1}{2}\frac{1+\sqrt{1-\Omega^2(E)}}{1-\Omega^2(E)} \\ u_0(E) &= \frac{1}{T^2} m\pi^2 a^2\big(1-\sqrt{1-\Omega^2(E)}\big) \\ \Omega(E) &= \frac{ET^2}{2m\pi^2 a^2} \\ u(x) &= \frac{1}{T^2} 2m\pi^2 x^2 \\ s(x) &= \frac{x}{a}\sqrt{1+\frac{x^2}{a^2}}. \end{aligned} \tag{43}$$

Therefore,

$$u(x) - Es(x) = \frac{1}{\gamma(E)} V(x) - u_0(E) \tag{44}$$

is the isochronic potential with the period $T(E) = T\sqrt{\gamma(E)}$ (it depends on E because E is now a parameter entering the 'effective' potential (44)). Note that $|E| < \frac{2m\pi^2 a^2}{T^2}$ and the potential (44) is confining in this region.

Consider now the set of potentials $V_k(x), k = 1, \ldots, N$, each of the form given by equation (15), satisfying in addition the following requirements:

(i) $\frac{c_k}{a_k} \equiv \sigma$ does not depend on k.
(ii) $T_k = n_k T$, where n_k are natural numbers while $T = \text{const}$ and let

$$H = \frac{\sum_{k=1}^N \left(\frac{p_k^2}{2m_k} + \frac{2m_k\pi^2}{n_k^2 T^2} x_k^2\right)}{\sum_{k=1}^N \frac{x_k}{a_k}\sqrt{1+\frac{x_k^2}{a_k^2}}}. \tag{45}$$

Separating the variables as above we obtain the system given by equation (33) with all potentials isochronic and corresponding periods equal $n_k T\sqrt{\gamma(E)}$ (note that due to the condition (i) $\gamma(E)$ is universal, i.e. does not depend on k). Therefore, all trajectories are closed for $|E| < \min\left(\frac{1}{n_k^2 T^2} 2m_k\pi^2 a_k^2\right)$. Our model is maximally superintegrable. Let us also note that action-angle variables $(\tilde{J}_k, \tilde{Q}_k)$ corresponding to the auxiliary Hamiltonian (35) are also action-angle variables for the original Hamiltonian (32) ((45) in the case under consideration). Consequently, the general procedure of constructing the additional integrals of motion, outlined above, applies here. Note that the whole procedure works also if one adds an arbitrary constant in the denominator on the right-hand side of (32). So, instead of (45) we can consider the Hamiltonian

$$H = \frac{\sum_{k=1}^N \left(\frac{p_k^2}{2m_k} + \frac{2m_k\pi^2}{n_k^2 T^2} x_k^2\right)}{1 + \sum_{k=1}^N \frac{x_k}{a_k}\sqrt{1+\frac{x_k^2}{a_k^2}}}. \tag{46}$$

This can be viewed as a deformation of a superintegrable system of harmonic oscillators; the underformed case is attained in the limit $a_k \to \infty, k = 1, \ldots, N$.

We finish this section with the remark that the problem of finding isochronic potentials of the form (39) is, in general, far from being trivial and remains open.

5. Conclusions

Let us briefly summarize our results. We have outlined the general method for constructing superintegrable systems for separated Hamiltonians. It should be stressed here that even for completely separated dynamics superintegrability is a highly nontrivial property and provides an exception rather than a rule. Our method allows us to find superintegrable systems for which the additional integrals are, in general, very complicated and cannot be computed analytically. In particular, they are not quadratic in momenta (and even more, not polynomials). This implies that alternative ways of separating variables in the HJ equation can only be achieved by general canonical transformations and not by point ones . It is rather clear that all previously found superintegrable models with separated Hamiltonians (at least in Cartesian coordinates) should be, more or less implicitly, embedded in our scheme (although an identification of the relevant function $F/\tilde{F}$ (see equations (8), (11)) generating given superintegrable Hamiltonian does not need to be straightforward).

Let us note that the procedure described in section 2 gives all potentials with oscillator-like energy spectrum in the WKB approximation. However, this property does not, in general, survive higher order corrections in $\hbar$. This means that the energy spectrum of corresponding superintegrable systems is degenerate also only in the WKB approximation; classical superintegrability does not survive quantization. This can also be seen from the complicated form of additional integrals of motion; in general, their quantum counterparts are unlikely to exist due to ordering problems.

The most interesting aspect of our analysis seems to be its application to Liouville-type Hamiltonians. As compared with the completely separated case these systems are distinguished by the explicit dependence of frequencies ω_k on total energy E; for example for the models (45) or (46) we get $\omega_k(E) = w_k/\sqrt{\gamma(E)}$ with $\gamma(E)$ given by (43). Liouville systems provide an important step towards general separable systems with quadratic integrals of motion i.e. Staeckel systems [25].

Acknowledgments

I would like to thank P Kosinski and P Maslanka for very interesting and enlightening discussions and suggestions. Unknown referees are acknowledged for remarks which were helpful in clarifying some points. The author was supported by KBN grant no 5 P03B06021.

References

[1] Adler M 1977 Some finite dimensional integrable systems and their scattering behavior *Comm. Mod. Phys.* **55** 195–230

[2] Arnold V I 1980 *Mathematical Methods in Classical Mechanics* (New York: Springer)

[3] Ballesteros A, Herranz F J, Santander M and Sanz-Gil T 2003 Maximal superintegrability on N-dimensional curved spaces *J. Phys. A: Math. Gen.* **36** L93–9

[4] Calzada J A, del Olmo M A and Rodriguez M A 1997 Classical superintegrable $SO(p, q)$ Hamiltonian systems *J. Geom. Phys.* **23** 14

[5] Drach J 1935 *C. R. Acad. Sci., Paris* **200** 22

[6] Evans N W 1990 Superintegrability in classical mechanics *Phys. Rev.* A **41** 5666–76

[7] Evans N W 1991 Group theory of the Smorodinsky–Winternitz system *J. Math. Phys.* **32** 3369–75

[8] Evans N W 1990 Superintegrability of the Winternitz system *Phys. Lett.* A **147** 483–6

[9] Fokas A S and Lagerstrom P A 1980 Quadratic and cubic invariants in classical mechanics *J. Math. Anal. Appl.* **74** 325–41

[10] Fris I, Mandrosov V, Smorodinsky J, Uhlir M and Winternitz P 1965 On higher-order symmetries in quantum mechanics *Phys. Lett.* **16** 354–6

[11] Goldstein H 1990 *Classical Mechanics* (Reading, MA: Addison-Wesley)

[12] Gonera C, Kosinski P and Maslanka P 2001 Superintegrable models of Winternitz type *Phys. Lett.* A **289** 66–8

[13] Gonera C and Majewski M 2001 Note on the algebraic structure of superintegrable systems *Acta Phys. Pol.* B **32** 1167

[14] Gonera C 1998 On the superintegrability of Calogero–Moser–Sutherland model *J. Phys. A: Math. Gen.* **31** 4465

[15] Gonera C 2002 More about generalized maximally superintegrable systems of Winternitz type *Preprint* hep-th/0207182 unpublished

[16] Gravel S and Winternitz P 2002 *J. Math. Phys.* **43** 5902

[17] Gravel S 2003 Hamiltonians separable in Cartesian coordinates and third-order integrals of motion *Preprint* math-ph/0302028

[18] Holt C R 1982 Construction of new integrable Hamiltonians in two degrees of freedom *J. Math. Phys.* **23** 1037–46

[19] Kalnins E G, Kress J M and Winternitz P 2002 Superintegrability in a two-dimensional space of non-constant curvature *J. Math. Phys.* **43** 970–83

[20] Karlovini M and Rosquist K 2000 A unified treatment of cubic invariants at fixed and arbitrary energiries *J. Math. Phys.* **41** 370–84

[21] Kalnins E G, Miller W Jr and Pogosyan G S 2001 Completeness of multiseparability in two-dimensional constant curvature space *J. Phys. A: Math. Gen.* **34** 4705–20

[22] Landau L and Lifshitz E 1976 *Mechanics* (Oxford: Pergamon)

[23] Makarov A, Smorodinsky J, Valiev Kh and Winternitz P 1967 A systematic search for non-relativistic systems with dynamical symmetries *Nuovo Cimento* A **52** 1061–84

[24] Onofri E and Pauri M 1978 Search for periodic Hamiltonian flows: a generalized Bertrand's theorem *J. Math. Phys.* **19** 1850

[25] Perelomov A M 1990 *Integrable Systems of Classical Mechanics and Lie Algebras* (Basle: Birkhauser)

[26] Ranada M F 1997 Superintegrable $n = 2$ systems, quadratic constants of motion and the potentials of Drach *J. Math. Phys.* **38** 4165–78

[27] Ranada M F and Santander M 1999 Superintegrable systems on the two-dimensional sphere S^2 and the hyperbolic plane H^2 *J. Math. Phys.* **40** 5026–57

[28] Rodriguez M A and Winternitz P 2002 Quantum superintegrability and exact solvability in n dimensions *J. Math. Phys.* **43** 1309–22

[29] Sheftel M B, Tempesta P and Winternitz P 2001 Superintegrable systems in quantum mechanics and classical Lie theory *J. Math. Phys.* **42** 659–73

[30] Tempesta P, Turbiner A V and Winternitz P 2001 Exact solvability of superintegrable systems *J. Math. Phys.* **42** 4248–57

[31] Tsiganov A V 2000 The Drach superintegrable systems *J. Phys. A: Math. Gen.* **33** 7407

[32] Winternitz P, Smorodinsky J, Uhlir M and Fris I 1996 Symmetry groups in classical and quantum mechanics *Yad. Fiz.* **4** 625–35

Winternitz P, Smorodinsky J, Uhlir M and Fris I 1967 *Sov. J. Nucl. Phys.* **4** 444–50 (Engl. Transl.)

[33] Wojciechowski S 1983 Superintegrability of the Calogero–Moser system *Phys. Lett.* A **95** 279–81